

A semilinear equation involving the fractional Laplacian in \mathbb{R}^n

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Abstract

In this paper, we consider the semilinear equation involving the fractional Laplacian in the Euclidian space \mathbb{R}^n :

$$(-\Delta)^{\alpha/2}u(x) = f(x_n)u^p(x), \quad x \in \mathbb{R}^n \quad (1)$$

in the subcritical case with $1 < p < \frac{n+\alpha}{n-\alpha}$. Instead of carrying out direct investigations on pseudo-differential equation (1), we first seek its equivalent form in an integral equation as below:

$$u(x) = \int_{\mathbb{R}^n} G_{\infty}(x, y) f(y_n) u^p(y) dy, \quad (2)$$

where $G_{\infty}(x, y)$ is the Green's function associated with the fractional Laplacian in \mathbb{R}^n . Exploiting the *method of moving planes in integral forms*, we are able to derive the nonexistence of positive solutions for (2) in the subcritical case. Hence the same conclusion is true for (1).

Key words *The fractional Laplacian, equivalence, Green's function, Kelvin transform, method of moving planes in integral forms, nonexistence of positive solutions, subcritical case.*

1 Introduction

The fractional Laplacian is a nonlocal operator defined as

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha}P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy, \quad (3)$$

where α can be any real number between 0 and 2, and P.V. stands for the Cauchy principle value.

Besides (3), the fractional Laplacian has two other equivalent definitions. One is using the *extension method* introduced by Caffarelli and Silvestre in [CS], and the other is by the Fourier transform:

$$\widehat{(-\Delta)^{\alpha/2}u}(\xi) = |\xi|^\alpha \hat{u}(\xi),$$

with \hat{u} denoting the Fourier transform of u . This operator is well defined in \mathcal{S} , the Schwartz space of rapidly decreasing C^∞ functions in \mathbb{R}^n , and it can be extended to the distributions in the space

$$\mathcal{L}_{\alpha/2} = \{u \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty\}$$

by

$$\langle (-\Delta)^{\alpha/2}u, \phi \rangle = \int_{\mathbb{R}^n} u (-\Delta)^{\alpha/2}\phi dx, \text{ for all } \phi \in C_0^\infty(\mathbb{R}^n).$$

For any domain $\Omega \subset \mathbb{R}^n$ and for a given $g \in L_{loc}^1(\Omega)$, we say that $u \in \mathcal{L}_{\alpha/2}$ is a solution to the problem

$$(-\Delta)^{\alpha/2}u(x) = g(x), \quad x \in \Omega$$

if and only if

$$\int_{\mathbb{R}^n} u (-\Delta)^{\alpha/2}\phi dx = \int_{\mathbb{R}^n} g(x) \phi(x) dx, \text{ for all } \phi \in C_0^\infty(\Omega). \quad (4)$$

In this paper, we only consider solutions in the distributional sense as given in (4).

We start by investigating the nonlocal equation with a specific nonlinearity in \mathbb{R}^n :

$$(-\Delta)^{\alpha/2}u(x) = x_n^2 u^p(x), \quad x \in \mathbb{R}^n, \quad (5)$$

in the subcritical case with $1 < p < \frac{n+\alpha}{n-\alpha}$.

Then we will deal with the problem assuming a more general form:

$$(-\Delta)^{\alpha/2}u(x) = f(x_n) u^p(x), \quad x \in \mathbb{R}^n, \quad (6)$$

where $f(x_n)$ is a positive increasing function.

For both pseudo-differential problems (5) and (6), we want to obtain the nonexistence of positive solutions u .

The main idea of the proof is as follows.

First, we prove that (5) is equivalent to the integral equation

$$u(x) = \int_{\mathbb{R}^n} G_\infty(x, y) y_n^2 u^p(y) dy, \quad (7)$$

where

$$G_\infty(x, y) = \frac{A_{n,\alpha}}{|x - y|^{n-\alpha}}$$

is the Green's function in \mathbb{R}^n .

Theorem 1 *Assume that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$. If u is a nonnegative solution of (5), then u satisfies the integral equation (7), and vice versa.*

Thanks to the equivalence between (5) and (7), in order to verify that (5) admits no positive solutions, it suffices to show that same conclusion holds for the integral equation (7). Utilizing the *method of moving planes in integral forms*, we verify

Theorem 2 *Assume that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$. If u is a nonnegative solution of (7) when $1 < p < \frac{n+\alpha}{n-\alpha}$, then $u(x) \equiv 0$.*

As an immediate consequence, we have the following

Corollary 1 *Assume that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$. If u is a nonnegative solution of (5) when $1 < p < \frac{n+\alpha}{n-\alpha}$, then $u(x) \equiv 0$.*

For (6), similarly, we obtain the equivalence in the first place.

Theorem 3 *Assume that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$ and f is an increasing positive function. If u is a nonnegative solution of (6), then u satisfies the integral equation*

$$u(x) = \int_{\mathbb{R}^n} G_\infty(x, y) f(y_n) u^p(y) dy, \quad (8)$$

and vice versa.

Then we employ the *method of moving planes* to show that

Theorem 4 *Assume that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$ and f is an increasing positive function. If u is a nonnegative solution of (8) when $1 < p < \frac{n+\alpha}{n-\alpha}$, then $u(x) \equiv 0$.*

Instantly, we arrive at

Corollary 2 *Assume that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$ and f is an increasing positive function. If u is a nonnegative solution of (6) when $1 < p < \frac{n+\alpha}{n-\alpha}$, then $u(x) \equiv 0$.*

The significance of such results on the nonexistence of global positive solutions lies in the fact that it serves as an important ingredient in obtaining a prior estimate for solutions of a corresponding family of nonlocal equations on bounded domains in the Euclidean space or on Riemannian manifolds. For more articles concerning the Liouville type theorem and integral equations, please see [CDL], [CFY], [CL2], [LZ], [RFB], [ZW], [ZCCY] and the references therein.

The paper is organised as follows: we present the main results in the introduction. The second section is devoted to the proof of Theorem 1 and Theorem 2. In the third section, we verify Theorem 3 and Theorem 4.

2 Nonexistence of positive solution for

$$(-\Delta)^{\alpha/2} u(x) = x_n^2 u^p(x)$$

2.1 Equivalence between the integral equation and the pseudo-differential equation

In [Ku], the Green's function of the operator $(-\Delta)^{\alpha/2}$ with the Dirichlet condition on the unit ball B_1 is obtained as:

$$G_1(x, y) = \frac{A_{n,\alpha}}{s^{\frac{n-\alpha}{2}}} \left[1 - B_{n,\alpha} \frac{1}{(t+s)^{\frac{n-2}{2}}} \int_0^{\frac{s}{t}} \frac{(s-tb)^{\frac{n-2}{2}}}{b^{\alpha/2}(1+b)} db \right], x, y \in B_1,$$

where $s = |x - y|^2$, $t = (1 - |x|^2)(1 - |y|^2)$ and $A_{n,\alpha}$ and $B_{n,\alpha}$ are constants relying on n and α .

Then we can write the Green's function on B_R as

$$\begin{aligned}
& G_R(x, y) \\
&= \frac{1}{R^{n-\alpha}} G_1\left(\frac{x}{R}, \frac{y}{R}\right) \\
&= \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} \left[1 - B_{n,\alpha} \frac{1}{\left(1 + \frac{s_R}{t_R}\right)^{\frac{n-2}{2}}} \int_0^{\frac{s_R}{t_R}} \frac{\left(\frac{s_R}{t_R} - b\right)^{\frac{n-2}{2}}}{b^{\alpha/2}(1+b)} db \right], \quad (9)
\end{aligned}$$

with $s_R = \frac{|x-y|^2}{R^2}$ and $t_R = (1 - |\frac{x}{R}|^2)(1 - |\frac{y}{R}|^2)$.

Let

$$v_R(x) = \int_{B_R} G_R(x, y) y_n^2 u^p(y) dy.$$

Since $u \in L_{loc}^\infty(\mathbb{R}^n)$, it is obvious that for each $R > 0$, $v_R(x)$ is well-defined and satisfies

$$\begin{cases} (-\Delta)^{\alpha/2} v_R(x) = x_n^2 u^p(x), & x \in B_R, \\ v_R(x) = 0, & x \notin B_R. \end{cases} \quad (10)$$

Let $w_R(x) = u(x) - v_R(x)$ and we have

$$\begin{cases} (-\Delta)^{\alpha/2} w_R(x) = 0, & x \in B_R, \\ w_R(x) \geq 0, & x \notin B_R. \end{cases} \quad (11)$$

To continue, we need the *maximum principle* for factional Laplacians.

Lemma 2.1 ([Si]) *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, and let u be a lower-semicontinuous function in $\overline{\Omega}$ such that $(-\Delta)^{\alpha/2} u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u \geq 0$ in \mathbb{R}^n .*

Applying Lemma 2.1 to (11), we arrive at

$$w_R(x) \geq 0, \quad x \in B_R.$$

Thus

$$w_R(x) \geq 0, \quad x \in \mathbb{R}^n.$$

Sending $R \rightarrow \infty$ in (9) and we obtain the Green's function $G_\infty(x, y)$ in \mathbb{R}^n :

$$G_\infty(x, y) = \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}}.$$

Meanwhile,

$$v_R(x) \rightarrow v(x) := \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} y_n^2 u^p(y) dy,$$

and

$$w_R(x) \rightarrow w(x) := u(x) - v(x).$$

It follows from (11) that

$$\begin{cases} (-\Delta)^{\alpha/2} w(x) = 0, & x \in \mathbb{R}^n, \\ w(x) \geq 0, & x \in \mathbb{R}^n. \end{cases} \quad (12)$$

Lemma 2.2 ([ZCCY]) *Every α -harmonic function bounded either above or below in all of \mathbb{R}^n for $n \geq 2$ must be constant.*

The above *Liouville theorem* for fractional Laplacians implies that

$$w(x) \equiv C \geq 0, \quad x \in \mathbb{R}^n.$$

Here and below C stands for nonnegative constants of different values in various line. Now it's obvious that $v(x)$ is well-defined. To establish the equivalence, we need to show that $C = 0$. Indeed, if $C > 0$, then for each fixed $x \in \mathbb{R}^n$,

$$\begin{aligned} \infty > u(x) &\geq v(x) \\ &= \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} y_n^2 u^p(y) dy \\ &\geq \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} y_n^2 C^p dy \\ &\geq C \int_{\mathbb{R}^n \setminus D} \frac{y_n^2}{|x-y|^{n-\alpha}} dy \\ &\geq C \int_{\mathbb{R}^n \setminus D} \frac{dy}{|x-y|^{n-\alpha}} \\ &= \infty, \end{aligned}$$

with $D = \{y \in \mathbb{R}^n \mid |y_n| < 1\}$. The contradiction above implies that $w = 0$. Therefore,

$$u(x) = v(x) = \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} y_n^2 u^p(y) dy.$$

Next we prove that if $u(x)$ solves the integral equation (7), it also solves the differential equation. For any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\langle (-\Delta)^{\frac{\alpha}{2}} u, \phi \rangle &= \langle \int_{\mathbb{R}^n} G_\infty(x, y) y_n^2 u^p(y) dy, (-\Delta)^{\frac{\alpha}{2}} \phi(x) \rangle \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} G_\infty(x, y) y_n^2 u^p(y) dy \right\} (-\Delta)^{\frac{\alpha}{2}} \phi(x) dx \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} G_\infty(x, y) (-\Delta)^{\frac{\alpha}{2}} \phi(x) dx \right\} y_n^2 u^p(y) dy \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \delta(x - y) \phi(x) dx \right\} y_n^2 u^p(y) dy \\
&= \int_{\mathbb{R}^n} \phi(y) y_n^2 u^p(y) dy \\
&= \langle y_n^2 u^p, \phi \rangle.
\end{aligned}$$

Hence $u(x)$ satisfies (5). This proves Theorem 1.

2.2 Nonexistence of positive solutions for the integral equation

We will use the Kelvin type transform and the method of moving planes to derive the nonexistence of positive solutions of (7) under the assumption that $u \in \mathcal{L}_{\alpha/2} \cap L_{loc}^\infty(\mathbb{R}^n)$.

With no global integrability assumptions on the solution of (7), we cannot carry out even the first step in the method of moving planes. To overcome this difficulty, we turn to \bar{u} , the Kelvin type transform of u , which has the desired integrability at infinity.

For $z^0 = (z^{0'}, 0) \in (\mathbb{R}^{n-1}, \mathbb{R})$, let

$$\bar{u}(x) = \frac{1}{|x - z^0|^{n-\alpha}} u \left(\frac{x - z^0}{|x - z^0|^2} + z^0 \right) \quad (13)$$

be the Kelvin type transform of u centered at z^0 . Apparently, \bar{u} is integrable near infinity.

Through an elementary calculation we have

$$\begin{aligned}
\bar{u}(x) &= \frac{1}{|x - z^0|^{n-\alpha}} \int_{\mathbb{R}^n} G_\infty \left(\frac{x - z^0}{|x - z^0|^2} + z^0, y \right) y_n^2 u^p(y) dy \\
&= \int_{\mathbb{R}^n} G_\infty(x, y) \frac{y_n^2 \bar{u}^p(y)}{|y - z^0|^\beta} dy,
\end{aligned} \quad (14)$$

for all $x \in \mathbb{R}^n \setminus B_\epsilon(z^0)$ and $\epsilon > 0$, where $\beta = 4 + (n - \alpha)(\tau - p) \geq 0$ and $\tau = \frac{n+\alpha}{n-\alpha}$.

Now we carry out the method of moving planes on a nonnegative solution \bar{u} of (14). Our goal is to show that \bar{u} is symmetric about the line passing through z^0 and parallel to the x_n axis. Such symmetry enables us to derive that u is independent of its first $(n - 1)$ th variables x_1, \dots, x_{n-1} , and consequently obtain that

$$u = u(x_n).$$

The fact that the value of $u(x)$ is determined by its x_n th variable only will lead to a contradiction with the finiteness of the integral

$$\int_{\mathbb{R}^n} G_\infty(x, y) y_n^2 u^p(y) dy.$$

By then it's easy to see that $u(x)$ must be trivial.

We begin the proof by introducing some notations. For a given real number λ , let the moving plane be

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}.$$

Let

$$\Sigma_\lambda = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 < \lambda\},$$

and

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of the point x about T_λ . Set

$$u_\lambda(x) = u(x^\lambda) \text{ and } w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x).$$

The argument will be presented in two parts. In the first part, we begin moving the plane T_λ from the neighbourhood of $x_1 = -\infty$. We want to show that for λ sufficiently negative,

$$w_\lambda(x) \geq 0, \text{ a.e. in } \Sigma_\lambda. \tag{15}$$

This provides a starting point for the moving of the planes. As long as (15) holds, we can keep moving the planes to the right until it reaches a limiting position $\lambda = z_1^0$. Going through a similar argument, one can move T_λ from the positive infinity to the left and show that

$$w_\lambda(x) \leq 0, \text{ a.e. in } \Sigma_\lambda, \tag{16}$$

for any $\lambda \geq z_1^0$. Combining (15) and (16), it's trivial to obtain the symmetry of $u_\lambda(x)$ about the plane $T_{z_1^0}$, i.e.

$$w_{z_1^0} \equiv 0, \text{ a.e. in } \Sigma_{z_1^0}. \quad (17)$$

This achieves the goal in part two.

Here is the detailed proof.

Step 1. Start moving the planes from $-\infty$ to the right as long as (15) holds.

For any $\epsilon > 0$, define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \setminus B_\epsilon((z^0)^\lambda) \mid w_\lambda(x) < 0\}.$$

We show that for λ sufficiently negative, Σ_λ^- must be measure zero.

By (14), we have

$$\begin{aligned} \bar{u}(x) &= \int_{\Sigma_\lambda} G_\infty(x, y) \frac{y_n^2 \bar{u}^p(y)}{|y - z^0|^\beta} dy + \int_{\Sigma_\lambda} G_\infty(x, y^\lambda) \frac{y_n^2 \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} dy, \\ \bar{u}(x^\lambda) &= \int_{\Sigma_\lambda} G_\infty(x^\lambda, y) \frac{y_n^2 \bar{u}^p(y)}{|y - z^0|^\beta} dy + \int_{\Sigma_\lambda} G_\infty(x^\lambda, y^\lambda) \frac{y_n^2 \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} dy. \end{aligned}$$

Then

$$\begin{aligned} &\bar{u}(x) - \bar{u}_\lambda(x) \\ &= \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \frac{y_n^2 \bar{u}^p(y)}{|y - z^0|^\beta} dy \\ &+ \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x^\lambda, y^\lambda)] \frac{y_n^2 \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} dy \\ &= \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{y_n^2 \bar{u}^p(y)}{|y - z^0|^\beta} - \frac{y_n^2 \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} \right] dy. \quad (18) \end{aligned}$$

By the Mean Value Theorem, for sufficiently negative values of λ and

$x \in \Sigma_\lambda^-$, we have

$$\begin{aligned}
0 &< \bar{u}(x) - \bar{u}_\lambda(x) \\
&= \int_{\Sigma_\lambda} y_n^2 [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} \right] dy \\
&= \int_{\Sigma_\lambda^-} y_n^2 [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} \right] dy \\
&\quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} y_n^2 [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} \right] dy \\
&\leq \int_{\Sigma_\lambda^-} y_n^2 [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} \right] dy \\
&= \int_{\Sigma_\lambda^-} y_n^2 [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{\bar{u}^p(y) - \bar{u}_\lambda^p(y)}{|y - z^0|^\beta} \right. \\
&\quad \left. + \bar{u}_\lambda^p(y) \left[\frac{1}{|y - z^0|^\beta} - \frac{1}{|y^\lambda - z^0|^\beta} \right] \right] dy \\
&\leq \int_{\Sigma_\lambda^-} y_n^2 [G_\infty(x, y) - G_\infty(x^\lambda, y)] \frac{\bar{u}^p(y) - \bar{u}_\lambda^p(y)}{|y - z^0|^\beta} dy \\
&\leq p \int_{\Sigma_\lambda^-} y_n^2 G_\infty(x, y) \frac{\bar{u}^{p-1}(y)}{|y - z^0|^\beta} [\bar{u}(y) - \bar{u}_\lambda(y)] dy \\
&\leq \int_{\Sigma_\lambda^-} \frac{C}{|x - y|^{n-\alpha}} \frac{\bar{u}^{p-1}(y)}{|y - z^0|^{\beta-2}} (\bar{u}(y) - \bar{u}_\lambda(y)) dy. \tag{19}
\end{aligned}$$

To continue, we need the following lemma.

Lemma 2.3 *(An equivalent form of the Hardy-Littlewood-Sobolev inequality) Assume $0 < \alpha < n$ and $\Omega \subset \mathbb{R}^n$. Let $g \in L^{\frac{np}{n+\alpha p}}(\Omega)$ for $\frac{n}{n-\alpha} < p < \infty$. Define*

$$Tg(x) := \int_{\Omega} \frac{1}{|x - y|^{n-\alpha}} g(y) dy.$$

Then

$$\|Tg\|_{L^p(\Omega)} \leq C(n, p, \alpha) \|g\|_{L^{\frac{np}{n+\alpha p}}(\Omega)}. \tag{20}$$

The proof of this lemma is standard and can be found in [CL1] or [CL2].

Let Ω be any domain that is a positive distance away from z^0 . Since u is locally bounded, we have

$$\int_{\Omega} \left[\frac{\bar{u}^{p-1}(y)}{|y - z^0|^{\beta-2}} \right]^{\frac{n}{\alpha}} dy < \infty. \quad (21)$$

For any $q > \frac{n}{n-\alpha}$, applying the Hardy-Littlewood-Sobolev inequality (20) and Hölder inequality to (19) yields

$$\begin{aligned} \|w_{\lambda}\|_{L^q(\Sigma_{\lambda}^-)} &\leq C \left\| \frac{\bar{u}^{p-1}}{|y - z^0|^{\beta-2}} w_{\lambda} \right\|_{L^{\frac{nq}{n+\alpha q}}(\Sigma_{\lambda}^-)} \\ &\leq C \left\| \frac{\bar{u}^{p-1}}{|y - z^0|^{\beta-2}} \right\|_{L^{\frac{n}{\alpha}}(\Sigma_{\lambda}^-)} \|w_{\lambda}\|_{L^q(\Sigma_{\lambda}^-)}. \end{aligned} \quad (22)$$

When N is sufficiently large, (21) indicates that for $\lambda \leq -N$,

$$C \left\{ \int_{\Sigma_{\lambda}^-} \left[\frac{\bar{u}^{p-1}}{|y - z^0|^{\beta-2}} \right]^{\frac{n}{\alpha}} dy \right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \quad (23)$$

A combination of (22) and (23) gives

$$\|w_{\lambda}\|_{L^q(\Sigma_{\lambda}^-)} = 0.$$

Therefore Σ_{λ}^- must be measure zero, that is,

$$w_{\lambda}(x) \geq 0, \quad \text{a.e. } x \in \Sigma_{\lambda}. \quad (24)$$

Step 2. Move the plane to the limiting position to derive symmetry.

Inequality (24) serves as a starting point to move the planes T_{λ} . As long as (24) is valid, we will continue moving the planes to the right until the limiting position. Define

$$\lambda_0 = \sup\{\lambda \leq z_1^0 \mid w_{\rho}(x) \geq 0, \rho \leq \lambda, \forall x \in \Sigma_{\rho}\}.$$

We prove

$$\lambda_0 \geq z_1^0 - \epsilon \quad (25)$$

using the contradiction argument. If (25) is not true, or, $\lambda_0 < z_1^0 - \epsilon$, then we are able to show that $\bar{u}(x)$ is symmetric about the plane T_{λ_0} , i.e.

$$w_{\lambda_0} \equiv 0, \quad \text{a.e. in } \Sigma_{\lambda_0}. \quad (26)$$

If (26) does not hold, then for such $\lambda_0 < z_1^0 - \epsilon$,

$$w_{\lambda_0} > 0, \text{ a.e. in } \Sigma_{\lambda_0}.$$

This enables us to move the plane even further to the right. More rigorously, there exists a $\zeta > 0$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \zeta)$,

$$w_\lambda(x) \geq 0, \text{ a.e. in } \Sigma_\lambda.$$

This contradicts the definition of λ_0 .

From inequality (22) it follows that

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C \left\{ \int_{\Sigma_\lambda^-} \left[\frac{\bar{u}^{p-1}(y)}{|y - z^0|^{\beta-2}} \right]^{\frac{n}{\alpha}} dy \right\}^{\frac{\alpha}{n}} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \quad (27)$$

When ζ is sufficiently small, for all $\lambda \in [\lambda_0, \lambda_0 + \zeta)$,

$$C \left\{ \int_{\Sigma_\lambda^-} \left[\frac{\bar{u}^{p-1}(y)}{|y - z^0|^{\beta-2}} \right]^{\frac{n}{\alpha}} dy \right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \quad (28)$$

We will give the proof of the inequality above later. For now, from (27) and (28), we can deduce that $\|w_\lambda\|_{L^q_{\Sigma_\lambda^-}} = 0$. Hence Σ_λ^- must be measure zero.

And for $\lambda > \lambda_0$, we have

$$w_\lambda(x) \geq 0, \text{ a.e. in } \Sigma_\lambda.$$

This is a contradiction with the definition of λ_0 . Therefore (26) holds. So far, we have verified that if $\lambda_0 < z_1^0 - \epsilon$ for any $\epsilon > 0$, then

$$\bar{u}(x) \equiv \bar{u}_{\lambda_0}(x), \text{ a.e. in } \Sigma_{\lambda_0}.$$

Furthermore, due to the singularity of \bar{u} at z^0 , \bar{u} is also singular at $(z^0)^{\lambda_0}$. It cannot be true given that z^0 is the only singularity of \bar{u} from (13). This proves that

$$\lambda_0 \geq z_1^0 - \epsilon. \quad (29)$$

Together with the arbitrariness of $\epsilon > 0$, it implies that

$$w_{z_1^0}(x) \geq 0, \text{ a.e. in } \Sigma_{z_1^0}.$$

Similarly, we can move the plane from near $x_1 = +\infty$ to the left and show that

$$w_{z_1^0}(x) \leq 0, \quad \text{a.e. in } \Sigma_{z_1^0}. \quad (30)$$

Therefore,

$$w_{z_1^0}(x) \equiv 0, \quad \text{a.e. in } \Sigma_{z_1^0}. \quad (31)$$

Now we prove inequality (28). For any positive small η and ϵ , for R sufficiently large it holds

$$\left(\int_{(\mathbb{R}^n \setminus B_\epsilon((z^0)^\lambda)) \setminus B_R} \left(\frac{\bar{u}^{p-1}(y)}{|y - z^0|^{\beta-2}} \right)^{\frac{n}{\alpha}} dy \right)^{\frac{\alpha}{n}} < \eta. \quad (32)$$

For any fixed R large we show that the measure of $\Sigma_\lambda^- \cap B_R$ is sufficiently small for λ close to λ_0 . Actually, we have

$$w_{\lambda_0}(x) > 0 \quad (33)$$

in the interior of $\Sigma_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0})$.

Indeed, if (33) is violated, then there exists some point $x_0 \in \Sigma_{\lambda_0}$ such that $u(x_0) = u_{\lambda_0}(x_0)$. From (18) we have

$$\begin{aligned} 0 &= \bar{u}(x_0) - \bar{u}_{\lambda_0}(x_0) \\ &= \int_{\Sigma_{\lambda_0}} y_n^2 [G_\infty(x_0, y) - G_\infty(x_0^{\lambda_0}, y)] \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_{\lambda_0}^p(y)}{|y^{\lambda_0} - z^0|^\beta} \right] dy. \end{aligned}$$

And further

$$u(x) > u_{\lambda_0}(x), \quad \forall x \in \Sigma_{\lambda_0}.$$

This obviously contradicts the fact that $w_{\lambda_0}(x) \geq 0$ in Σ_{λ_0} . Hence (33) is true.

On the other hand, the measure of $(\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap B_R$ can also be made as small as we want. Combining this with (32), (28) follows easily.

For any $\gamma > 0$, let

$$E_\gamma = \{(x \in \Sigma_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0})) \cap B_R \mid w_{\lambda_0}(x) > \gamma\},$$

$$F_\gamma = ((\Sigma_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0})) \cap B_R) \setminus E_\gamma.$$

Obviously,

$$\lim_{\gamma \rightarrow 0} \mu(F_\gamma) = 0.$$

For $\lambda > \lambda_0$, let

$$D_\lambda = (((\Sigma_\lambda \setminus B_\epsilon((z^0)^\lambda)) \setminus (\Sigma_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0})))) \cap B_R.$$

Then

$$((\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap B_R) \subset ((\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap E_\gamma) \cup F_\gamma \cup D_\lambda. \quad (34)$$

For λ close to λ_0 , both the measures of D_λ and $(\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap E_\gamma$ are close to zero. In fact, for any $x \in (\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap E_\gamma$, we have

$$w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x) = \bar{u}_\lambda(x) - \bar{u}_{\lambda_0}(x) + \bar{u}_{\lambda_0}(x) - \bar{u}(x) < 0.$$

Therefore,

$$\bar{u}_{\lambda_0}(x) - \bar{u}_\lambda(x) > w_{\lambda_0}(x) > \gamma.$$

Further,

$$((\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap E_\gamma) \subset G_\gamma \equiv \{x \in B_R \mid u_{\lambda_0}(x) - u_\lambda(x) > \gamma\}. \quad (35)$$

By the well-known Chebyshev inequality, we have

$$\begin{aligned} \mu(G_\gamma) &\leq \frac{1}{\gamma^{p+1}} \int_{G_\gamma} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} dx \\ &\leq \frac{1}{\gamma^{p+1}} \int_{B_R} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} dx. \end{aligned} \quad (36)$$

For each fixed γ , as λ goes to λ_0 , the right hand side of the above inequality goes to zero. This implies that the measure of $(\Sigma_\lambda^- \setminus B_\epsilon((z^0)^\lambda)) \cap B_R$ can also be made arbitrarily small.

This completes the proof of (31).

Since we can choose any direction that is perpendicular to the x_n -axis as the x_1 direction, by showing (31) or \bar{u} is symmetric about the plane $T_{z_1^0}$, we have actually shown that $\bar{u}(x)$ is rotationally symmetric about the line parallel to the x_n -axis and passing through z^0 for $1 < p < \frac{n+\alpha}{n-\alpha}$. Now, for any two points X^1 and X^2 with $X^i = (x^{i'}, x_n) \in R^{n-1} \times R$, $i = 1, 2$. Let z^0 be the projection of $\bar{X} = \frac{X^1 + X^2}{2}$ on $\partial \mathbb{R}_+^n = \{x = (x', x_n) \in R^n \mid x_n = 0\}$. Set $Y^i = \frac{X^i - z^0}{|X^i - z^0|^2} + z^0$, $i = 1, 2$. From the above arguments, one can easily see $\bar{u}(Y^1) = \bar{u}(Y^2)$, hence $u(X^1) = u(X^2)$. This implies that u is independent of x_1, \dots, x_{n-1} , which, in fact, contradicts the finiteness of the integral

$$\int_{\mathbb{R}^n} G_\infty(x, y) y_n^2 u^p(y) dy.$$

If $u(x) = u(x_n)$ solves

$$u(x) = \int_{\mathbb{R}^n} G_\infty(x, y) y_n^2 u^p(y) dy, \quad (37)$$

then for each fixed $x = (x', x_n)$, let $r = |x' - y'|$, $t = |x_n - y_n|$, $s = r/t$ and we have

$$\begin{aligned} +\infty > u(x_n) &= \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x - y|^{n-\alpha}} y_n^2 u^p(y) dy \\ &= \int_R y_n^2 u^p(y_n) \int_{\mathbb{R}^{n-1}} \frac{dy'}{|x - y|^{n-\alpha}} dy_n \\ &= \int_R y_n^2 u^p(y_n) \int_0^\infty \frac{C r^{n-2} dr}{|r^2 + t^2|^{\frac{n-\alpha}{2}}} dy_n \\ &= \int_R y_n^2 u^p(y_n) t^{\alpha-1} \int_0^\infty \frac{s^{n-2} ds}{(s^2 + 1)^{\frac{n-\alpha}{2}}} dy_n \\ &\geq C \int_R y_n^2 u^p(y_n) |x_n - y_n|^{\alpha-1} dy_n \end{aligned} \quad (38)$$

$$\begin{aligned} &\geq C \int_{R_0} y_n^2 u^p(y_n) \left(\frac{y_n}{2}\right)^{\alpha-1} dy_n \\ &= C \int_{R_0}^\infty y_n^{1+\alpha} u^p(y_n) dy_n \end{aligned} \quad (39)$$

for sufficiently large R_0 . The finiteness of (39) indicates that there exists a sequence $\{y_n^i\}_{i \rightarrow \infty}$ such that

$$u^p(y_n^i) (y_n^i)^{\alpha+2} \rightarrow 0. \quad (40)$$

For any fixed $x = (0, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, let $x_n = 2R$ be large. From (38) we deduce that

$$\begin{aligned} +\infty > u(x_n) &\geq C \int_1^2 u^p(y_n) y_n^2 |x_n - y_n|^{\alpha-1} dy_n \\ &\geq C x_n^{\alpha-1}. \end{aligned} \quad (41)$$

Together with (38), (41) indicates that for $x_n = 2R$ sufficiently large,

$$\begin{aligned} u(x_n) &\geq C \int_{\frac{R}{2}}^R [C x_n^{\alpha-1}]^p y_n^2 R^{\alpha-1} dy_n \\ &= C_{p,\alpha} R^{(\alpha-1)(p+1)+3} \\ &:= D_{p,\alpha} x_n^{(\alpha-1)(p+1)+3} \end{aligned} \quad (42)$$

Repeating the substitution process above for another m times and setting $x_n = 2R$, we arrive at

$$\begin{aligned} u(x_n) &\geq D(m, p, \alpha) x_n^{3+3p+3p^2+\dots+3p^m+(p^{m+1}+p^m+\dots+1)(\alpha-1)} \\ &= D(m, p, \alpha) x_n^{\frac{p^{m+1}(p-p\alpha-3)+2+\alpha}{1-p}}. \end{aligned} \quad (43)$$

For $1 \leq \alpha < 2$, it's easy to see that as $x_n \rightarrow \infty$

$$u^p(x_n) x_n^{\alpha+2} \rightarrow \infty.$$

This contradicts (40). For $0 < \alpha < 1$, for sufficiently large m , it holds that

$$u^p(x_n) x_n^{\alpha+2} \geq D(m, p, \alpha) x_n^{\tau(p)} \geq D(m, p, \alpha) > 0, \quad (44)$$

for all x_n sufficiently large with

$$\begin{aligned} &\tau(p) \\ &= 3 + 3p + \dots + 3p^m + (p^{m+1} + p^m + \dots + 1)(\alpha - 1) + \alpha + 2 \end{aligned} \quad (45)$$

$$= \frac{p^{m+2}(p - p\alpha - 3) + 2p + \alpha}{1 - p} + 2 \geq 0. \quad (46)$$

Again this is a contradiction with (40). Hence we declare that (37) admits no positive solution.

To prove (46), it suffices to show that for m sufficiently large and $\alpha \in (0, 1)$,

$$\tau'(p) > 0, \forall p \in (1, \frac{n + \alpha}{n - \alpha}),$$

since (45) shows that $\tau(1) > 0$. Indeed,

$$\tau'(p) = \frac{p^{m+1} [m(p(\alpha - 1) + 3)(p - 1) + 6p - 3\alpha p + 2\alpha p^2 - 2p^2 - 6] + 2 + \alpha}{(1 - p)^2}, \quad (47)$$

and $(p(\alpha - 1) + 3)(p - 1) > 0$ for $n \geq 2$ and $0 < \alpha < 1$. Therefore, it's true that when m is large enough, $\tau'(p) > 0$.

This completes the proof of Theorem 2.

3 Nonexistence of positive solution for

$$(-\Delta)^{\alpha/2} u(x) = f(x_n) u^p(x)$$

3.1 Equivalence between the integral equation and the pseudo-differential equation

Let

$$v(x) = \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} f(y_n) u^p(y) dy. \quad (48)$$

Going through exactly the same reasoning as that in section 2.1, we can show that if $u(x)$ satisfies

$$(-\Delta)^{\alpha/2} u(x) = f(x_n) u^p(x), \quad x \in \mathbb{R}^n, \quad (49)$$

then

$$u(x) - v(x) \equiv C \geq 0.$$

We can see that $C = 0$. Otherwise, for every given $x \in \mathbb{R}^n$,

$$\begin{aligned} \infty > u(x) &\geq v(x) \\ &= \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} f(y_n) u^p(y) dy \\ &\geq \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} f(y_n) C^p dy \\ &\geq C \int_{\mathbb{R}^n \setminus D} \frac{dy}{|x-y|^{n-\alpha}} \\ &= \infty, \end{aligned}$$

with $D = \{y \in \mathbb{R}^n \mid |y_n| < 1\}$. The contradiction shows that $w = 0$. Therefore,

$$u(x) = v(x) = \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} f(y_n) u^p(y) dy. \quad (50)$$

In the distributional sense, a solution for (50) is also a solution for (49).

Actually, for any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\langle (-\Delta)^{\frac{\alpha}{2}} u, \phi \rangle &= \langle \int_{\mathbb{R}^n} G_\infty(x, y) f(y_n) u^p(y) dy, (-\Delta)^{\frac{\alpha}{2}} \phi(x) \rangle \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} G_\infty(x, y) f(y_n) u^p(y) dy \right\} (-\Delta)^{\frac{\alpha}{2}} \phi(x) dx \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} G_\infty(x, y) (-\Delta)^{\frac{\alpha}{2}} \phi(x) dx \right\} f(y_n) u^p(y) dy \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \delta(x - y) \phi(x) dx \right\} f(y_n) u^p(y) dy \\
&= \int_{\mathbb{R}^n} \phi(y) f(y_n) u^p(y) dy \\
&= \langle f(y_n) u^p, \phi \rangle.
\end{aligned}$$

Hence a solution for the integral equation satisfies the differential equation as well. This proves Theorem 3.

3.2 Nonexistence of positive solutions for the integral equation

Performing the same Kelvin transform as defined in (13) on u that solves

$$u(x) = \int_{\mathbb{R}^n} G_\infty(x, y) f(y_n) u^p(y) dy, \quad (51)$$

then \bar{u} , the Kelvin transform of u , satisfies

$$\begin{aligned}
\bar{u}(x) &= \frac{1}{|x - z^0|^{n-\alpha}} \int_{\mathbb{R}^n} G_\infty\left(\frac{x - z^0}{|x - z^0|^2} + z^0, y\right) f(y_n) u^p(y) dy \\
&= \int_{\mathbb{R}^n} G_\infty(x, y) \frac{1}{|x - y|^{n-\alpha}} \frac{f\left(\frac{y_n}{|y - z^0|^2}\right) \bar{u}^p(y)}{|y - z^0|^\beta} dy, \quad (52)
\end{aligned}$$

for all $x \in \mathbb{R}^n \setminus B_\epsilon(z^0)$ and $\epsilon > 0$, where $\beta = (n - \alpha)(\tau - p) \geq 0$ and $\tau = \frac{n+\alpha}{n-\alpha}$. By (51), we have

$$\begin{aligned}
& \bar{u}(x) - \bar{u}_\lambda(x) \\
&= \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^p(y)}{|y-z^0|^\beta} dy \\
&+ \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x^\lambda, y^\lambda)] \frac{f(\frac{y_n}{|y^\lambda-z^0|^2}) \bar{u}_\lambda^p(y)}{|y^\lambda-z^0|^\beta} dy \\
&= \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^p(y)}{|y-z^0|^\beta} - \frac{f(\frac{y_n}{|y^\lambda-z^0|^2}) \bar{u}_\lambda^p(y)}{|y^\lambda-z^0|^\beta} \right] dy.
\end{aligned} \tag{53}$$

Use the same T_λ , Σ_λ and x^λ as introduced at the beginning of section 2.1. Set

$$u_\lambda(x) = u(x^\lambda) \text{ and } w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x).$$

As before, we work on a starting point for the moving of the planes in step 1 by proving that for λ sufficiently negative,

$$w_\lambda(x) \geq 0, \text{ a.e. in } \Sigma_\lambda. \tag{54}$$

In step 2, one shall see that the moving plane will not stop until it arrives at the limiting position $\lambda = z_1^0$. Then by moving the planes from the very right to the left, we will have

$$w_\lambda(x) \leq 0, \text{ a.e. in } \Sigma_\lambda, \tag{55}$$

for any $\lambda \geq z_1^0$. Together with (54), it yields that

$$w_{z_1^0} \equiv 0, \text{ a.e. in } \Sigma_{z_1^0}. \tag{56}$$

Here comes the proof in details.

Step 1. Start moving the planes from $-\infty$ to the right as long as (54) holds.

For any $\epsilon > 0$, define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \setminus B_\epsilon((z^0)^\lambda) \mid w_\lambda(x) < 0\}.$$

We show that for λ sufficiently negative, Σ_λ^- must be measure zero.

It follows from the Mean Value Theorem and (53) that for $x \in \Sigma_\lambda^-$ and λ sufficiently negative, we have

$$\begin{aligned}
0 &< \bar{u}(x) - \bar{u}_\lambda(x) \\
&= \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^p(y)}{|y-z^0|^\beta} - \frac{f(\frac{y_n}{|y^\lambda-z^0|^2}) \bar{u}_\lambda^p(y)}{|y^\lambda-z^0|^\beta} \right] dy \\
&= \int_{\Sigma_{\lambda^-}} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^p(y)}{|y-z^0|^\beta} - \frac{f(\frac{y_n}{|y^\lambda-z^0|^2}) \bar{u}_\lambda^p(y)}{|y^\lambda-z^0|^\beta} \right] dy \\
&\quad + \int_{\Sigma_\lambda \setminus \Sigma_{\lambda^-}} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^p(y)}{|y-z^0|^\beta} - \frac{f(\frac{y_n}{|y^\lambda-z^0|^2}) \bar{u}_\lambda^p(y)}{|y^\lambda-z^0|^\beta} \right] dy \\
&\leq \int_{\Sigma_{\lambda^-}} [G_\infty(x, y) - G_\infty(x^\lambda, y)] \left[\frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^p(y)}{|y-z^0|^\beta} - \frac{f(\frac{y_n}{|y^\lambda-z^0|^2}) \bar{u}_\lambda^p(y)}{|y^\lambda-z^0|^\beta} \right] dy \\
&\leq \int_{\Sigma_{\lambda^-}} [G_\infty(x, y) - G_\infty(x^\lambda, y)] f(\frac{y_n}{|y-z^0|^2}) \frac{\bar{u}^p(y) - \bar{u}_\lambda^p(y)}{|y-z^0|^\beta} dy \\
&\leq p \int_{\Sigma_{\lambda^-}} G_\infty(x, y) f(\frac{y_n}{|y-z^0|^2}) \frac{\bar{u}^{p-1}(y)}{|y-z^0|^\beta} [\bar{u}(y) - \bar{u}_\lambda(y)] dy \\
&= \int_{\Sigma_{\lambda^-}} \frac{C}{|x-y|^{n-\alpha}} f(\frac{y_n}{|y-z^0|^2}) \frac{\bar{u}^{p-1}(y)}{|y-z^0|^\beta} (\bar{u}(y) - \bar{u}_\lambda(y)) dy. \tag{57}
\end{aligned}$$

Let Ω be any domain that is a positive distance away from z^0 . Since u is locally bounded, we have

$$\int_{\Omega} \left[\frac{f(\frac{y_n}{|y-z^0|^2}) \bar{u}^{p-1}(y)}{|y-z^0|^\beta} \right]^{\frac{n}{\alpha}} dy < C \int_{\Omega} \left[\frac{\bar{u}^{p-1}(y)}{|y-z^0|^\beta} \right]^{\frac{n}{\alpha}} dy < \infty. \tag{58}$$

For any $q > \frac{n}{n-\alpha}$, applying the Hardy-Littlewood-Sobolev inequality (20) and Hölder inequality to (57) yields

$$\begin{aligned}
\|w_\lambda\|_{L^q(\Sigma_{\lambda^-})} &\leq C \|f(\frac{y_n}{|y^\lambda-z^0|^2}) \frac{\bar{u}^{p-1}}{|y-z^0|^\beta} w_\lambda\|_{L^{\frac{nq}{n+\alpha q}}(\Sigma_{\lambda^-})} \\
&\leq C \|f(\frac{y_n}{|y^\lambda-z^0|^2}) \frac{\bar{u}^{p-1}}{|y-z^0|^\beta}\|_{L^{\frac{n}{\alpha}}(\Sigma_{\lambda^-})} \|w_\lambda\|_{L^q(\Sigma_{\lambda^-})} \\
&\leq C \|\frac{\bar{u}^{p-1}}{|y-z^0|^\beta}\|_{L^{\frac{n}{\alpha}}(\Sigma_{\lambda^-})} \|w_\lambda\|_{L^q(\Sigma_{\lambda^-})}. \tag{59}
\end{aligned}$$

When N is sufficiently large, (58) indicates that for $\lambda \leq -N$,

$$C \left\{ \int_{\Sigma_\lambda^-} \left[\frac{\bar{u}^{p-1}}{|y - z^0|^\beta} \right]^{\frac{n}{\alpha}} dy \right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \quad (60)$$

A combination of (59) and (60) gives

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0.$$

Therefore Σ_λ^- must be measure zero, that is,

$$w_\lambda(x) \geq 0, \quad \text{a.e. } x \in \Sigma_\lambda. \quad (61)$$

Step 2. Move the plane to the limiting position to derive symmetry.

Let

$$\lambda_0 = \sup\{\lambda \leq z_1^0 \mid w_\rho(x) \geq 0, \rho \leq \lambda, \forall x \in \Sigma_\rho\}.$$

We show that

$$\lambda_0 = z_1^0$$

and $\bar{u}(x)$ is symmetric about the plane T_{λ_0} , i.e.

$$w_{\lambda_0} \equiv 0, \quad \text{a.e. in } \Sigma_{\lambda_0}. \quad (62)$$

Despite that β takes a different value here, the rest of the proof will be the same as that for $(-\Delta)^{\alpha/2}u(x) = x_n^2 u^p(x)$, because $f(\frac{y_n}{|y-z^0|^2})\bar{u}^{p-1}(y)$ can be controlled by $\bar{u}^{p-1}(y)$ for $0 < |\lambda - z_1^0| < \epsilon$ with fixed ϵ .

The arbitrariness of the choice for the x_1 direction contributes to $\bar{u}(x)$'s rotational symmetry about the line parallel to the x_n -axis and passing through z^0 . This implies that u relies on its x_n th variable only. However, if $u(x) = u(x_n)$ solves

$$u(x) = \int_{\mathbb{R}^n} G_\infty(x, y) f(y_n) u^p(y) dy, \quad (63)$$

then for each fixed $x = (x', x_n)$, let $r = |x' - y'|$, $t = |x_n - y_n|$, $s = r/t$ and

it gives

$$\begin{aligned}
+\infty > u(x_n) &= \int_{\mathbb{R}^n} \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} f(y_n) u^p(y_n) dy \\
&= \int_R f(y_n) u^p(y_n) \int_{\mathbb{R}^{n-1}} \frac{dy'}{|x-y|^{n-\alpha}} dy_n \\
&= \int_R f(y_n) u^p(y_n) \int_0^\infty \frac{C r^{n-2} dr}{|r^2+t^2|^{\frac{n-\alpha}{2}}} dy_n \\
&= \int_R f(y_n) u^p(y_n) t^{\alpha-1} \int_0^\infty \frac{s^{n-2} ds}{(s^2+1)^{\frac{n-\alpha}{2}}} dy_n \\
&\geq C \int_R f(y_n) u^p(y_n) |x_n - y_n|^{\alpha-1} dy_n \tag{64} \\
&\geq C \int_{R_0} f(y_n) u^p(y_n) \left(\frac{y_n}{2}\right)^{\alpha-1} dy_n \\
&= C \int_{R_0}^\infty f(y_n) y_n^{\alpha-1} u^p(y_n) dy_n \tag{65}
\end{aligned}$$

for sufficiently large R_0 . The finiteness of (65) indicates that there exists a sequence $\{y_n^i\}_{i \rightarrow \infty}$ as $i \rightarrow \infty$, such that

$$f(y_n^i) u^p(y_n^i) (y_n^i)^\alpha \rightarrow 0. \tag{66}$$

Due to the monotonicity of $f(y_n)$, we further deduce that

$$u^p(y_n^i) (y_n^i)^\alpha \rightarrow 0. \tag{67}$$

For any fixed $x = (0, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, let $x_n = 2R$ be large. From (64) we deduce that

$$\begin{aligned}
+\infty > u(x_n) &\geq C \int_1^2 u^p(y_n) f(y_n) |x_n - y_n|^{\alpha-1} dy_n \\
&\geq C x_n^{\alpha-1}. \tag{68}
\end{aligned}$$

Together with (64), (68) indicates that for $x_n = 2R$ sufficiently large,

$$\begin{aligned}
u(x_n) &\geq C \int_{\frac{R}{2}}^R [C x_n^{\alpha-1}]^p f(y_n) R^{\alpha-1} dy_n \\
&= C_{p,\alpha} R^{(\alpha-1)(p+1)+1} \\
&:= D_{p,\alpha} x_n^{\alpha+\alpha p-p} \tag{69}
\end{aligned}$$

Going through the substitution process above for another m times and setting $x_n = 2R$, it gives

$$u(x_n) \geq D(m, p, \alpha) x_n^{\alpha + \alpha p + \alpha p^2 + \dots + \alpha p^{m+1} - p^{m+1}}. \quad (70)$$

For $1 \leq \alpha < 2$, it's easy to see that as $x_n \rightarrow \infty$

$$u^p(x_n) x_n^\alpha \rightarrow \infty.$$

This contradicts (67). For $0 < \alpha < 1$, for sufficiently large m , it holds that

$$u^p(x_n) x_n^\alpha \geq D(m, p, \alpha) x_n^{\tau(p)} \geq D(m, p, \alpha) > 0, \quad (71)$$

for all x_n sufficiently large with

$$\begin{aligned} & \tau(p) \\ = & \alpha + \alpha p + \alpha p^2 + \dots + \alpha p^{m+1} - p^{m+1} + \alpha \end{aligned} \quad (72)$$

$$= \frac{\alpha - p^{m+2} - (\alpha - 1) p^{m+3}}{1 - p} \geq 0. \quad (73)$$

Again this is a contradiction with (67). Hence we declare that (51) admits no positive solution. To prove (73), it suffices to show that for m sufficiently large and $\alpha \in (0, 1)$,

$$\tau'(p) > 0, \forall p \in (1, \frac{n + \alpha}{n - \alpha}),$$

since (72) shows that $\tau(1) > 0$. Indeed,

$$\tau'(p) = \frac{p^{m+1} [m(p(\alpha - 1) + 1)(p - 1) + 4p^2(\alpha - 1) + 3p(2 - \alpha) - 2] - \alpha}{(1 - p)^2}, \quad (74)$$

and $(p(\alpha - 1) + 1)(p - 1) > 0$ for $n \geq 2$ and $0 < \alpha < 1$. Therefore, it's true that when m is large enough, $\tau'(p) > 0$.

This completes the proof of Theorem 4.

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